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SOME ORBITAL ELEMENTS USEFUL IN SPACE TRAJECTORY CALCULATIONS

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ABSTRACT

A set of orbital elements is described which is applicable when the motion is hyperbolic and even rectilinear, such as the motion associated with a lunar landing. This set provides a convenient description of miss distance and furnishes simple relationships for many complex geometrical problems in lunar and interplanetary flight. The use of these elements with others which are described enables the computation of a variety of differential corrections.

[1]This paper presents the results of one phase of research carried out at the Jet Propulsion Laboratory, California Institute of Technology, under Contract No. NASw-6, sponsored by the National Aeronautics and Space Administration.

I. INTRODUCTION

Space-flight research characteristically involves the calculation of innumerable trajectories, some of which are reference trajectories and others variations of reference trajectories. The orbital elements given here can be used to determine differential corrections or changes from the standard trajectory without the necessity of computing varied trajectories, providing a welcome reduction in computation time.

As an example of their use, these orbital elements can be applied in the determination of a trajectory which impacts the target in a given way. If the target were on the Earth's surface, the miss distance would be described in terms of range and azimuth error, and a typical differential correction would be the partial derivative of range error with respect to speed at burnout. However, if the target is the Moon and a vertical impact is desired, some other measure of miss distance has to be used. By analogy with the terrestrial target calculations, two components are necessary. For this purpose, with a lunar target, use is made of two components of $\vec{B}$, the impact parameter vector, which has magnitude b and lies in the orbital plane. It is directed from the focus perpendicularly to the incoming asymptote of the osculating hyperbola, evaluated at a time when the vehicle is near the target. The elliptic case is not treated.

Having described the miss distance in terms of orbital elements, one can obtain analytically the approximate changes in these elements due to changes in the initial osculating conic referenced to the Earth. Of course, it is not necessary to obtain these changes analytically in order to use the elements as a convenient

measure of miss distance. When it is known how the miss components vary with changes in the initial conditions, either approximately or by a numerical solution of the varied equations of motion, it is possible to arrive at any desired kind of impact.

In this paper the general theory and methods of computation are given. The results will be discussed in subsequent papers. The use of orbital elements for differential corrections is described by Eckert and Brouwer (Ref. 1) and S. Herrick with others of Aeronutronic (Ref. 2).

II. THE $\vec{B}$ SET OF ELEMENTS

For near-rectilinear hyperbolic motion the $\vec{B}$ set of elements has been found to be a well-behaved set of parameters that produces no singularities, provided the motion does not become parabolic. The variations of this set with respect to changes in the initial coordinates are surprisingly linear. This set is being used exclusively at the Laboratory to design trajectories (Ref. 3).

The $\vec{B}$ vector, which forms the basis of the $\vec{B}$ set, is defined as a vector originating at the focus of hyperbola and directed to the incoming asymptote (see Fig. 1). Analytically,

$$\vec{B} = b\left(\frac{b}{c}\vec{P} + \frac{a}{c}\vec{Q}\right) \tag{1}$$

where

$$b = \frac{c_1}{\sqrt{c_3}} \tag{2}$$

with c_1 the angular momentum constant, c_3 the vis viva, or twice the total energy; and

$$c = \sqrt{a^2 + b^2} \tag{3}$$

The other symbols have the usual meaning. When defined in this way, $\vec{B}$ can be thought of as the vector miss which would occur if the target had no mass, although this is not strictly correct because the mass of the target does influence the osculating elements.

The direction of the incoming asymptote is specified by the unit vector $\vec{S}$:

$$\vec{S} = -\frac{a}{c}\vec{P} + \frac{b}{c}\vec{Q} \tag{4}$$

It can be shown that in attempts to hit a target, $\vec{S}$ will have approximately the same orientation for all search trajectories. Since $\vec{B}$ is perpendicular to $\vec{S}$, it follows that $\vec{B}$ can be represented approximately by two numbers, or components. The components chosen are along unit vectors $\vec{T}$ and $\vec{R}$, perpendicular to $\vec{S}$ and to each other. $\vec{T}$ lies in the equatorial, ecliptic, or other convenient fixed reference plane, and indicates whether the miss is to the right or left (direct or retrograde). The only condition for the fixed reference plane is that it is not perpendicular to the asymptote. The other vector, $\vec{R}$, indicates the up or down component of miss. The two components are $\vec{B} \cdot \vec{T}$ and $\vec{B} \cdot \vec{R}$, called m_1 and m_2.

If $\vec{S}$ is specified by two angles ϕ_S and θ_S, defined by

$$\sin \phi_S = S_z \tag{5}$$

$$\cos \phi_S = \sqrt{S_x^2 + S_y^2} \tag{6}$$

$$\cos \theta_S \cos \phi_S = S_x \tag{7}$$

$$\sin \theta_S \cos \phi_S = S_y \tag{8}$$

then $\vec{T}$ is given by

$$T_x = \sin \theta_S \tag{9}$$

$$T_y = -\cos \theta_S \tag{10}$$

$$T_z = 0 \tag{11}$$

$$\vec{R} = \vec{T} \times \vec{S} \tag{12}$$

A positive value for $\vec{B} \cdot \vec{T}$ indicates that the motion is direct; a positive value for $\vec{B} \cdot \vec{R}$ indicates that the motion is above the target.

If the approximation is used that $\vec{S}$ is constant for a given class of trajectories, and that the components m_1 and m_2 are linear functions of initial errors, then valuable information can be obtained analytically by calculating only one trajectory near the target. For example, it is possible by a relatively simple procedure to find a mapping between the miss components and the selenographic coordinates. Since the probability distribution for impacting can be easily determined in terms of the miss components (where linear theory is applicable), the distribution in terms of the selenographic coordinates may be obtained by the known mapping.

The mapping of miss components to selenographic coordinates depends essentially on two relations, shown in Fig. 2. The line OP is the path directed at the center of the target. This line also represents the asymptote of the osculating conic, which is very nearly parallel to the asymptotes of other conics which intersect the surface of the target. Consideration is now given to the case where the impact is not vertical and the impact parameter is along the $\vec{T}$ axis. Since the plane of the trajectory contains the original $\vec{S}$ vector, the loci of the impact points on the surface of the target (considered a sphere) must be a great circle. This is true whenever the direction of $\vec{B}$ is held fixed and the magnitude is varied.

This is the first relation. The other is that the angle between great circles at P is the same as the angle between the corresponding $\vec{B}$ vectors.

Another example of the application of this method is in searching for a trajectory which passes over the pole of the Moon. The conditions can be converted into miss components by first specifying the pericenter distance q; this is equivalent to specifying the altitude at closest approach. Then the required value of b is

$$b = \sqrt{q^2 - 2aq} \tag{13}$$

where a is negative for the hyperbola.

Since

$$b = \sqrt{m_1^2 + m_2^2} \tag{14}$$

the ratio of the components and their signs must be determined. This is found by projecting a unit vector in the direction of the pole, $\vec{O}$, on the $\vec{T}$ and $\vec{R}$ axes. Then

$$\frac{m_1}{b} = \frac{\vec{O} \cdot \vec{T}}{\sqrt{(\vec{O} \cdot \vec{T})^2 + (\vec{O} \cdot \vec{R})^2}} \tag{15}$$

$$\frac{m_2}{b} = \frac{\vec{O} \cdot \vec{R}}{\sqrt{(\vec{O} \cdot \vec{T})^2 + (\vec{O} \cdot \vec{R})^2}} \tag{16}$$

Thus a seemingly complex problem is solved by the use of three simple equations, (13), (15), and (16).

The six orbital elements that are used are the epoch of periapsis T, m_1, m_2, c_3, ϕ_S, and θ_S. To find the rectangular coordinates from these elements, c_1 is found from (14) and (2), $\vec{S}$ from (5), (7), and (8), $\vec{T}$ from (9) to (11), and $\vec{R}$ from (12), from which

$$\vec{B} = m_1 \vec{T} + m_2 \vec{R} \tag{17}$$

From (1) and (4), $\vec{P}$ and $\vec{Q}$ are found. If the epoch is given, Kepler's equation can be solved to find F, and the rectangular coordinates may be computed. Special techniques for solving Kepler's equation have been developed which apply when the eccentricity is close to 1.

III. THE w SET OF ELEMENTS

When the orbital plane is well determined, the w set of elements is used: T, c_1, c_3, w_1, w_2, and w_3. The w_i angles represent rotations about the $\vec{P}$, $\vec{Q}$, and $\vec{W}$ axes in a given order and have been defined so that they apply to finite rotations. Here it will be assumed that the rotations are infinitesinual.

The use of the rotational elements is similar to that described by Eckert and Brouwer (Ref. 2), who use rotations about the $\vec{P}$, $\vec{Q}$, and $\vec{W}$ axes. The rotations w_1, w_2, and w_3 are about the reference $\vec{P}$, $\vec{Q}$, and $\vec{W}$ axes. The other elements were chosen so that they would apply to elliptic motion with a large eccentricity, to parabolic, and to hyperbolic motion. For small eccentricities where the periapsis is not well determined, $-nT + w_3$ may be used as a variable, similar to Eckert and Brouwer. Since these elements must apply to motion where the eccentricity is close to 1, it was decided to eliminate the eccentricity as a variable because it is not well behaved. Also, a cannot be used because it is discontinuous. Instead,

$$c_3 = -\frac{\mu}{a} \tag{18}$$

is used, and is well behaved. The reason for the choice of T instead of M as an element follows if the parabolic case must be included. Thus T, c_1, c_3, and the w_i were chosen as orbital elements.

These elements are used in two different ways depending on whether the w_i are finite (as in guidance studies) or infinitesimal (as in differential correction computations). If the w_i are to be kept small, the standard $\vec{P}$, $\vec{Q}$, and $\vec{W}$ are

taken to be the osculating $\vec{P}$, $\vec{Q}$, and $\vec{W}$ along the reference trajectory. Then not only are the variations small, but the rotations are about "body-fixed" axes. In other words, if initially a w_1 rotation is performed, at a later time this will represent a rotation about the $\vec{P}$ axis of the later osculating conic.

IV. DIFFERENTIAL CORRECTIONS

A. Orbital-to-Rectangular Changes

The formulas which give the changes in inertial rectangular coordinates due to changes in the orbital elements will now be presented.

1. Changes in the w set. The w set is treated first, since some of its relations are further utilized in the transformation of the $\vec{B}$ set.

The equations for the hyperbola read

$$\vec{r} = a\vec{P}(\cosh F - e) + b\vec{Q} \sinh F \tag{19}$$

$$\dot{\vec{r}} = -\sqrt{-\mu a}\ \vec{P} \frac{\sinh F}{r} + \sqrt{\mu p}\ \frac{\vec{Q}\cosh F}{r} \tag{20}$$

Differentiating, and keeping t, the epoch at which the coordinates are observed, fixed, yields

$$\begin{aligned} d\vec{r} &= da\vec{P}(\cosh F - e) + d\vec{P}a(\cosh F - e) \\ &+ dFa\vec{P} \sinh F - dea\vec{P} + db\vec{Q} \sinh F \\ &+ d\vec{Q}b \sinh F + dFb\vec{Q} \cosh F \end{aligned} \tag{21}$$

$$\begin{aligned} d\dot{\vec{r}} &= da\ \frac{1}{2}\ \sqrt{\frac{\mu}{-a}}\ \vec{P} \frac{\sinh F}{r} - d\vec{P} \sqrt{-\mu a}\ \vec{P} \frac{\sinh F}{r} \\ &- dF\ \sqrt{-\mu a}\ \vec{P} \frac{\cosh F}{r} + dr\ \sqrt{-\mu a}\ \vec{P} \frac{\sinh F}{r^2} \\ &+ dp\ \frac{1}{2} \sqrt{\frac{\mu}{p}}\ \vec{Q} \frac{\cosh F}{r} + d\vec{Q} \sqrt{\mu p}\ \frac{\cosh F}{r} \\ &+ dF \sqrt{\mu p}\ \vec{Q} \frac{\sinh F}{r} - dr\ \sqrt{\mu p}\ \vec{Q} \frac{\cosh F}{r^2} \end{aligned} \tag{22}$$

These equations as they stand are not in terms of the orbital elements T, c_1, c_3, w_1, w_2, and w_3. Each will be expanded in turn. From the definition

$$da = -\frac{a}{c_3} dc_3 \tag{23}$$

$d\vec{P}$, $d\vec{Q}$, and $d\vec{W}$ are found from

$$d\vec{P} = \vec{Q}w_3 - \vec{W}w_2 \tag{24}$$

$$d\vec{Q} = \vec{W}w_1 - \vec{P}w_3 \tag{25}$$

$$d\vec{W} = \vec{P}w_2 - \vec{Q}w_1 \tag{26}$$

as can be seen from drawing a simple picture of any rotation. Equations (24) to (26) can be put in matrix form:

$$\begin{pmatrix} dP_x & dQ_x & dW_x \\ dP_y & dQ_y & dW_y \\ dP_z & dQ_z & dW_z \end{pmatrix} = \begin{pmatrix} P_x & Q_x & W_x \\ P_y & Q_y & W_y \\ P_z & Q_z & W_z \end{pmatrix} \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix} \tag{27}$$

This form is suitable if the rotations are finite, and the last matrix represents the derivatives of an orthogonal matrix representing the rotation. Also by direct differentiation,

$$de = \frac{1}{2e\mu^2}\left(2c_1c_3dc_1 + c_1^2dc_3\right) \tag{28}$$

$$db = \frac{dc_1}{\sqrt{|c_3|}} - \frac{1}{2}\frac{b}{c_3}dc_3 \tag{29}$$

$$dn = \frac{3}{2}\frac{\mu}{c_3}dc_3 \tag{30}$$

$$dp = \frac{2c_1dc_1}{\mu} \tag{31}$$

Incidentally, (23) to (31) also apply when the conic is an ellipse. Kepler's equation is differentiated:

$$dF = \frac{dn(t - T) - ndT - de \sinh F}{e \cosh F - 1} \tag{32}$$

where dn and de have already been given in terms of the orbital elements. Also,

$$dr = -da(e \cosh F - 1) - de\, a \cosh F - dFae \sinh F \tag{33}$$

The formulas for the elliptic case are so similar that they are not listed.

2. Changes in the $\vec{B}$ set. For this set, the division between parameters which determine the orientation (the w_i) and parameters which determine the other properties is lost. Most of the elements affect both the orientation and shape of the conic. However, Eqs. (21) and (22) hold. Since

$$\vec{P} = -\vec{S}\frac{a}{c} + \left(\frac{\vec{B}}{b}\right)\frac{b}{c} \tag{34}$$

and

$$\vec{Q} = \vec{S}\frac{b}{c} + \left(\frac{\vec{B}}{b}\right)\frac{a}{c} \tag{35}$$

the variations in $\vec{P}$ and $\vec{Q}$ can be derived if the right-hand side is differentiable. For the rectilinear case, when b approaches zero, the term in (35) involving $\vec{B}$ can cause difficulty. Since the rectilinear case presents some analytical problems, it will be fully treated. The nonrectilinear case is straightforward, and will not be analyzed in detail.

From Eqs. (28) and (31), de and dp are zero since e_1 is zero for the rectilinear case. Hence, the equations of motion (19) and (20) can be differentiated keeping e = 1 and p = 0:

$$\begin{aligned} d\vec{r} &= da\vec{P}(\cosh F - 1) + d\vec{P}a(\cosh F - 1) \\ &\quad + dFa\vec{P}\sinh F + db\vec{Q}\sinh F \end{aligned} \tag{36}$$

$$\begin{aligned} d\dot{\vec{r}} &= da\ \frac{1}{2}\sqrt{-\frac{\mu}{a}}\ \vec{P}\ \frac{\sinh F}{r} - d\vec{P}\ \sqrt{-\mu a}\ \frac{\sinh F}{r} \\ &\quad - dF\ \sqrt{-\mu a}\ \vec{P}\ \frac{\cosh F}{r} + dr\ \sqrt{-\mu a}\ \vec{P}\ \frac{\sinh F}{r^2} \end{aligned} \tag{37}$$

It should be noted that $\vec{Q}$ appears in Eq. (36) and has not been defined for the rectilinear case. It will be defined so that it depends on each variation in a natural way. First da and dF are given by Eqs. (23) and (32) as before. Differentiating (34) and evaluating it for the rectilinear case,

$$d\vec{P} = d\vec{S} + d\vec{B}\ \frac{1}{c} \tag{38}$$

$d\vec{Q}$ is not used.

$$\begin{aligned} dS_x &= -\sin\theta_s\cos\phi_s\,d\theta_s - \cos\theta_s\sin\phi_s d\phi_s \\ dS_y &= \cos\theta_s\cos\phi_s d\theta_s - \sin\theta_s\sin\phi_s\,d\phi_s \\ dS_z &= \cos\phi_s\,d\phi_s \end{aligned} \tag{39}$$

$$d\vec{B} = d(\vec{B}\cdot\vec{T})\ \vec{T} + d(\vec{B}\cdot\vec{R})\ \vec{R} \tag{40}$$

$\vec{T}$ and $\vec{R}$ are given unambiguously by Eqs. (5) to (12).

It now remains to define $\vec{Q}$. Since $d\vec{B}$ is contained in the $\vec{T}$, $\vec{R}$ plane from (40), it is perpendicular to $\vec{S}$, or to $\vec{P}$ which is equal to $\vec{S}$. Hence, there is no reason why $\vec{Q}$ should not be defined to be in the opposite direction to $d\vec{B}$. This will insure that $\vec{B}$ is a continuous function for the variation in any element. Hence,

$$\vec{Q} = -\frac{m_1}{\sqrt{m_1^2 + m_2^2}}\vec{T} - \frac{m_2}{\sqrt{m_1^2 + m_2^2}}\vec{R} \tag{41}$$

Since

$$b = \sqrt{m_1^2 + m_2^2} \tag{42}$$

$$db = \frac{m_1}{b}\,dm_1 + \frac{m_2}{b}\,dm_2 \tag{43}$$

This formula must be modified when b goes to zero, which is the rectilinear case. If it is assumed that dm_1 or dm_2 is positive, then

$$db = dm_1 \tag{44}$$

for an m_1 variation, and

$$db = dm_2 \tag{45}$$

for an m_2 variation. In general,

$$db = \sqrt{(dm_1)^2 + (dm_2)^2} \tag{46}$$

For the rectilinear case, c_1 is no longer an independent variable

$$dc_1 = db\sqrt{c_3} \tag{47}$$

B. Rectangular-to-Orbital Changes

1. Changes in the w set. It is a simple matter to find the changes in the angular momentum and energy constants due to changes in the rectangular coordinates:

$$c_1 = \left|\vec{r} \times \dot{\vec{r}}\right| = \sqrt{(\vec{r} \times \dot{\vec{r}}) \cdot (\vec{r} \times \dot{\vec{r}})} \tag{48}$$

$$dc_1 = \frac{\vec{r} \times \dot{\vec{r}} \cdot \left[d\vec{r} \times \dot{\vec{r}} + \vec{r} \times d\dot{\vec{r}}\right]}{\left|\vec{r} \times \dot{\vec{r}}\right|} \tag{49}$$

$$= \vec{W} \cdot \left[d\vec{r} \times \dot{\vec{r}} + \vec{r} \times d\dot{\vec{r}}\right] \tag{50}$$

For the rectilinear case $\vec{W}$ is determined by the perturbation, and Eq. (44) reduces to

$$dc_1 = \left|d\vec{r} \times \dot{\vec{r}} + \vec{r} \times d\dot{\vec{r}}\right| \tag{51}$$

which can also be derived from the fact that c_1 originally is zero. Since

$$c_3 = \dot{\vec{r}} \cdot \dot{\vec{r}} - \frac{2\mu}{|\vec{r}|} \tag{52}$$

$$dc_3 = 2\dot{\vec{r}} \cdot d\dot{\vec{r}} + \frac{2\mu\vec{r} \cdot d\vec{r}}{r^3} \tag{53}$$

The formulas for changes in a, e, p, and n, which depend only on c_1 and c_3, are unchanged. For the hyperbola from (33),

$$dF = -\frac{dr + da(e \cosh F - 1) + de(a \cosh F)}{ae \sinh F} \tag{54}$$

$$dr = \frac{\vec{r}}{r} \cdot d\vec{r} \tag{55}$$

$$dT = -\frac{1}{n}\left[de \sinh F + dF(e \cosh F - 1) - dn(t - T)\right] \quad (56)$$

The changes in orientation will next be given. Since

$$\vec{W} = \frac{\vec{r} \times \dot{\vec{r}}}{c_1} \quad (57)$$

$$d\vec{W} = \frac{d\vec{r} \times \dot{\vec{r}} + \vec{r} \times d\dot{\vec{r}}}{c_1} - \frac{\vec{W}}{c_1} dc_1 \quad (58)$$

From Eq. (26),

$$d\vec{W} \cdot \vec{P} = w_2 \quad (59)$$

$$d\vec{W} \cdot \vec{Q} = -w_1 \quad (60)$$

To find w_3, $\hat{v}$ is introduced:

$$\hat{v} = v + w_3 \quad (61)$$

The term $\hat{v}$ represents the polar angle that the vehicle makes with the reference $\vec{P}$ axis for infinitesimal rotations (see Fig. 3).

Hence,

$$dw_3 = d\hat{v} - dv \quad (62)$$

Since

$$\cos v = \frac{1}{e}\left(\frac{p}{r} - 1\right) \tag{63}$$

$$dv = \frac{1}{\sin v}\left[\frac{de}{e^2}\left(\frac{p}{r} - 1\right) - \frac{dp}{er} + \frac{dr\,p}{er^2}\right] \tag{64}$$

$$d\hat{v} = d\vec{r} \cdot \left[\frac{\vec{P}\sin v}{-r} + \frac{\vec{Q}\cos v}{r}\right] \tag{65}$$

This last formula can easily be derived by noting that $d\vec{r} \cdot \vec{P}$ and $d\vec{r} \cdot \vec{Q}$ are the components of $d\vec{r}$ on the $\vec{P}$ and $\vec{Q}$ axes. A change in position along the $\vec{P}$ axis results in a rotation of $-\sin v/r$, etc.

Thus the changes in the w set can be computed.

2. Changes in the $\vec{B}$ set, rectilinear case. The changes in c_1, c_3, T, etc. remain the same as in the w set, but the changes in m_1, m_2, ϕ_s, and θ_s must be computed. It is convenient to define a unit normal to $\vec{W}$, for each variation in the rectangular coordinates. Since

$$c_1\vec{W} = \vec{r} \times \dot{\vec{r}} \tag{66}$$

and $d\vec{W}$ is defined to be zero,

$$dc_1\vec{W} = d\vec{r} \times \dot{\vec{r}} + \vec{r} \times d\dot{\vec{r}} \tag{67}$$

or

$$\vec{W} = \frac{d\vec{r} \times \dot{\vec{r}} + \vec{r} \times d\dot{\vec{r}}}{\left| d\vec{r} \times \dot{\vec{r}} + \vec{r} \times d\dot{\vec{r}} \right|} \tag{68}$$

As an example for a variation in x,

$$\vec{W} = \frac{1}{\sqrt{\dot{y}^2 + \dot{z}^2}} \begin{pmatrix} 0 \\ -\dot{z} \\ \dot{y} \end{pmatrix} \tag{69}$$

$$\vec{B} = b(\vec{S} \times \vec{W}) \tag{70}$$

$$d\vec{B} = db(\vec{S} \times \vec{W}) \tag{71}$$

$$d(\vec{B} \cdot \vec{T}) = (d\vec{B}) \cdot \vec{T} \tag{72}$$

$$d(\vec{B} \cdot \vec{R}) = (d\vec{B}) \cdot \vec{R} \tag{73}$$

For the rectilinear or almost rectilinear case,

$$\vec{S} = \frac{\dot{\vec{r}}}{\dot{s}} \tag{74}$$

where

$$\dot{s} = \sqrt{\dot{\vec{r}} \cdot \dot{\vec{r}}} \tag{75}$$

This assumes that the vehicle is a sufficient distance from the focus that it is moving parallel to the asymptote. In practice, Eq. (74) will be an excellent approximation if the other conditions for the linear case are satisfied. Hence,

$$d\vec{S} = \frac{d\vec{r}}{\dot{s}} - \vec{S}\,\frac{\dot{\vec{r}} \cdot d\vec{r}}{\dot{s}^2} \qquad (76)$$

From (39),

$$d\phi_S = \frac{dS_Z}{\cos\phi_S} \qquad (77)$$

$$d\theta_S = \frac{dS_Y + \sin\theta_S \sin\phi_S \, d\phi_S}{\cos\theta_S \cos\phi_S} \qquad (78)$$

C. Differential Corrections

In the general problem of differential corrections one is concerned with the changes in the position and velocity coordinates of some point along the path due to changes in the orbital elements at a different point. For the two-body problem the path can be represented analytically and the deviations can be obtained rigorously. Even if the path is not exactly an ellipse, it may be possible to calculate changes in the path as if a conic solution applied. The free flight path of an ICBM differs so little from an ellipse that in this case the deviations may be computed on the basis of formulas that are derived from the two-body problem.

This program of error correction can be used only in the case where the actual path is almost a conic section and the elements chosen exhibit the following properties. Consider a reference trajectory at time t_0. One of its elements q_i (t_0) is subjected to a small perturbation δq_i (t_0), while all the other elements are kept fixed. One now computes the path from t_0 to a later time t_1, denoting the elements of the varied path by $\overline{q}_i$. If now

$$\overline{q}_i\,(t_1) - q_i\,(t_1) \simeq \delta q_i\,(t_0) \qquad i = 1 \cdots 6 \qquad (79)$$

and

$$\bar{q}_j (t_1) - q_j (t_1) \simeq 0 \qquad i \neq j \tag{80}$$

then the elements are well behaved and no special variational equations need to be solved since

$$\frac{\partial q_j (t_1)}{\partial q_i (t_0)} \simeq \delta_{ij} \tag{81}$$

The orbital elements that have been chosen exhibit this property for trajectories encountered in deep space exploration where eccentricities are not close to zero. It has been shown, however, that a lunar or interplanetary trajectory can be broken up into segments each of which is in a region of a dominating body. Let us examine in this light a trajectory which is meant to intersect with the Moon and calculate the differential corrections of errors at the Moon as a function of errors at injection.

Using the w set at injection (the orbital plane being well defined), one converts the various errors at injection into orbital elements using the osculating conic sections at injection. Since the region of influence of the Moon is a concentric sphere of 66,000 km radius, this is taken as the boundary between Earth and Moon domination. At this point the changes in the elements of the osculating conic sections referenced to the Earth have to be changed to those referenced to the Moon. This transformation is accomplished in three steps. The first step is to convert the orbital changes of the Earth-referenced osculating conic into variations in an Earth-referenced, nonrotating, cartesian coordinate system. The next step is to convert

the cartesian coordinate changes to a Moon-referenced cartesian system. If the two coordinate systems are parallel, it follows that

$$\vec{R}_{12} + \vec{r}_2 = \vec{r}_1$$
$$\dot{\vec{R}}_{12} + \dot{\vec{r}}_2 = \dot{\vec{r}}_1$$

where $\vec{R}_{12}$ is the position of the second center of attraction with respect to the first, $\vec{r}_2$ is the position of the vehicle with respect to the second center, $\vec{r}_1$ is the position of the vehicle with respect to the first center; and similarly for the velocities. Thus the variations

$$\delta\vec{r}_1 = \delta\vec{r}_2$$
$$\delta\dot{\vec{r}}_1 = \delta\dot{\vec{r}}_2$$

are continuous at the boundary between the two regions. The third step is to convert the Moon-centered cartesian system into changes in the osculating conic referenced to the Moon. The osculating conics used in the calculations are, of course, the ones at the boundary and consequently may differ materially from the initial as well as from the final values.

In the neighborhood of the target the $\vec{B}$ set is used. With this analytical method one obtains the partial derivatives of the miss components at the target with a precision of a few per cent. The errors incurred are due mainly to effects at the boundary of the two regions. Studies are under way to circumvent this problem.

V. CONCLUSIONS

The $\vec{B}$ set and the w set have proven to be very useful in the determination of differential corrections. The $\vec{B}$ set can be used for cases where the motion is hyperbolic, and the w set is used for all motions except rectilinear motion and nearly circular motion. The combination of the two sets may be employed to predict changes in the final conditions due to changes in the initial conditions. Also, corrections in the orbit can be made by the same technique.

The method of solving the equations of motion by variation of parameters may be derived from the equations given here. It is possible to obtain the effects of the perturbations on the two-body problem by treating the perturbations as incremental velocities. From the transformation of velocities to orbital elements (presented here for certain cases) the equations for the variation of parameters follow.

Another application of these elements is in guidance studies. Here the w set is employed for finite rotations. For this application it is useful to have the end conditions in terms of the $\vec{B}$ set represented by a polynomial as a function of the initial orbital elements of the w set. Here the varied trajectories are calculated numerically to find the polynomial.

The complete equations used, including the method of solution of Kepler's equations, will be given in a forthcoming report.

NOMENCLATURE

a	semimajor axis
A	rotation matrix defined by Eq. (20)
b	semiminor axis
$\vec{B}$	the impact parameter vector defined by Eq. (1)
c	defined for the hyperbola by Eq. (3)
c_1	the angular momentum constant $= r^2\dot{v}$
c_3	the <u>vis viva</u>, or total energy $= \dot{s}^2 - (2\mu/r)$
e	eccentricity
E	eccentric anomaly for an ellipse
F	corresponding anomaly for the hyperbola
m_1, m_2	miss components $= \vec{B} \cdot \vec{T}$ and $\vec{B} \cdot \vec{T}$, respectively
n	mean motion
$\vec{O}$	unit vector in the direction of the Moon's pole
p	semilatus rectum of the conic
$\vec{P}$	unit vector directed toward the periapsis
q	distance from focus to vertex of conic
$\vec{Q}$	unit vector in orbital plane normal to $\vec{P}$
r	distance from focus
$\vec{r}$	position vector
$\vec{R}$, $\vec{T}$	unit vectors perpendicular to $\vec{S}$ which are associated with the miss components, defined by Eqs. (9) to (12)

NOMENCLATURE (Cont'd)

$\dot{s}$	the speed
$\vec{S}$	unit vector in the direction of the incoming asymptote, defined by Eq. (4)
t	time, epoch
T	epoch of periapsis
v	true anomaly
$\hat{v}$	defined by Eq. (61)
w_1, w_2, w_3	establish the rotation of the $\vec{P}$, $\vec{Q}$, and $\vec{W}$ axes, defined by Eq. (20)
$\vec{W}$	the unit vector normal to the plane; used with $\vec{P}$ and $\vec{Q}$
μ	gravitational constant multiplied by the mass of the attracting center
ϕ_S, θ_S	angles which give orientation of $\vec{S}$, defined by Eqs. (5) to (8)

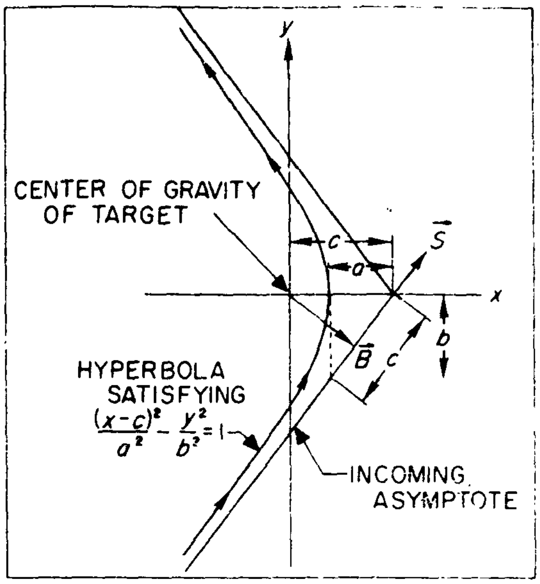


Fig. 1. Geometry of hyperbolic path near target

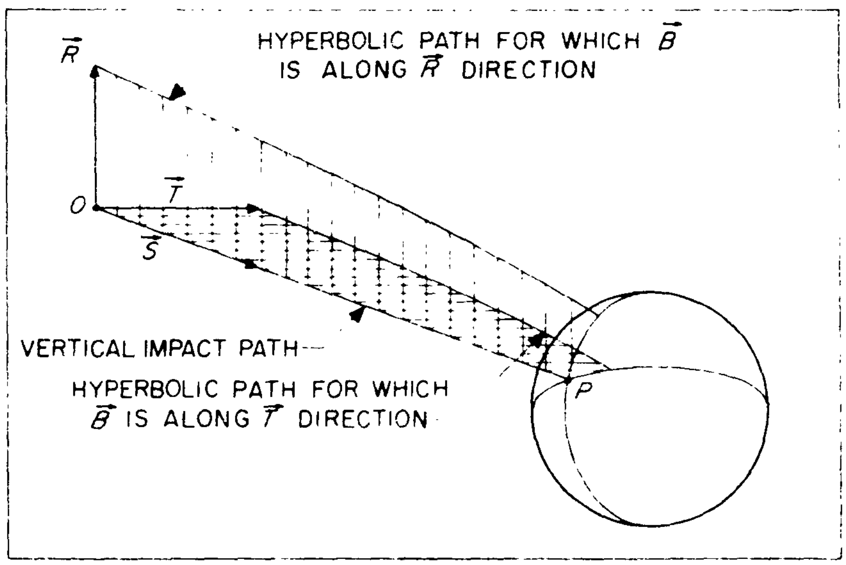
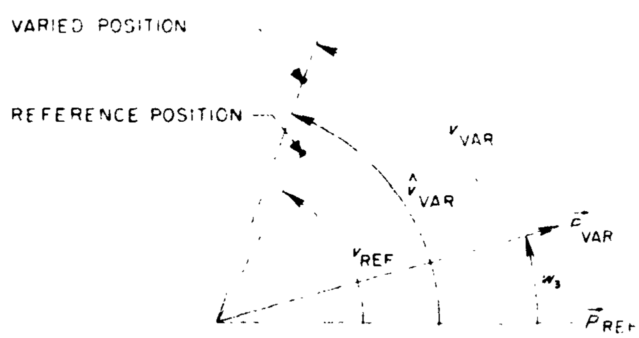


Fig. 2. Effect of variation of $\vec{B} \cdot \vec{T}$ and $\vec{B} \cdot \vec{R}$ on surface of target

Fig. 3. The relationship of v, $\hat{v}$, and w_3



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